priate value of the constant c (in the relation $\nu_T = ck^{\frac{1}{2}}\ell_m$) is

$$c = |\langle uv \rangle / k|^{\frac{1}{2}} \approx 0.55. \tag{10.42}$$

Use the relation $\mathcal{P} = \varepsilon$ to show

$$\varepsilon = c^3 k^{\frac{3}{2}} / \ell_m, \tag{10.43}$$

and hence

$$\nu_T = c^4 k^2 / \varepsilon. \tag{10.44}$$

Exercise 10.4 For the one-equation model applied to simple shear flow, express the production $\mathcal{P}$ in terms of k, ℓ_m and $\partial\langle U \rangle / \partial y$. Hence (taking $C_D = c^3$ in Eq. 10.37) show that the velocity scales u^* in the one-equation model and in the mixing-length model are related by

$$ck^{\frac{1}{2}} = \ell_m \left| \frac{\partial \langle U \rangle}{\partial y} \right| \left(\frac{\mathcal{P}}{\varepsilon} \right)^{-\frac{1}{2}}. \tag{10.45}$$

Show that the corresponding relation for a general flow is

$$ck^{\frac{1}{2}} = \ell_m \mathcal{S} \left(\frac{\mathcal{P}}{\varepsilon} \right)^{-\frac{1}{2}}, \tag{10.46}$$

(cf. Eq. 10.20).

10.4 The k-ε Model

10.4.1 Overview

The k-ε model belongs to the class of *two-equation models*, in which modelled transport equations are solved for two turbulence quantities—i.e., k and ε in the k-ε model. From these two quantities can be formed a lengthscale ($L = k^{\frac{3}{2}}/\varepsilon$), a timescale ($\tau = k/\varepsilon$), a quantity of dimension ν_T (k^2/ε), etc. As a consequence, two-equation models can be *complete*—flow-dependent specification such as $\ell_m(\mathbf{x})$ are not required.

The k-ε model is the most widely used complete turbulence model, and it is incorporated in most commercial CFD codes. As with all turbulence models, both the concepts and the details evolved over time; but Jones and Launder (1972) are appropriately credited with developing the "standard" k-ε model, with Launder and Sharma (1974) providing improved values of the model constants. Significant earlier contributions are due to Davidov (1961), Harlow and Nakayama (1968), Hanjalić (1970), and others cited by Launder and Spalding (1972).

In addition to the turbulent viscosity hypothesis, the k-ε model consists of:

(i) the modelled transport equation for k (which is the same as in the one-equation model, Eq. 10.41)

(ii) the modelled transport equation for ε (which is described below)

(iii) the specification of the turbulent viscosity as

$$\nu_T = C_\mu k^2/\varepsilon, \tag{10.47}$$

where $C_\mu = 0.09$ is one of five model constants.

If it is supposed that ν_T depends only on the turbulence quantities k and ε (independent of $\partial\langle U_i\rangle/\partial x_j$ etc.), then Eq. (10.47) is inevitable. The one-equation model implies the similar relation, $\nu_T = c^4 k^2/\varepsilon$ (see Exercise 10.3), and so the model constants are related by $c = C_\mu^{\frac{1}{4}}$.

In simple turbulent shear flow, the k-ε model yields

$$\frac{|\langle uv\rangle|}{k} = C_\mu^{\frac{1}{2}} \frac{\mathcal{P}}{\varepsilon}, \tag{10.48}$$

(see Exercise 10.5) so that the specification $C_\mu = 0.09 = (0.3)^2$ stems from the empirical observation $|\langle uv\rangle|/k \approx 0.3$ in regions where $\mathcal{P}/\varepsilon$ is close to unity.

The quantity $\nu_T\varepsilon/k^2$ plotted in Figs. 10.3 and 10.4 is a "measurement" of C_μ for channel flow and for the temporal mixing layer. As may be seen, $\nu_T\varepsilon/k^2$ is everywhere close to the value 0.09, except near the boundaries of the flows.

Exercise 10.5 Consider the k-ε model applied to a simple turbulent shear flow with $\mathcal{S} = \partial\langle U\rangle/\partial y$ being the only non-zero mean velocity gradient. Obtain the relations

$$\frac{|\langle uv\rangle|}{k} = C_\mu \frac{\mathcal{S}k}{\varepsilon}, \tag{10.49}$$

and

$$\frac{\mathcal{P}}{\varepsilon} = C_\mu \left(\frac{\mathcal{S}k}{\varepsilon}\right)^2, \tag{10.50}$$

and hence verify Eq. (10.48).

Show that $\langle uv \rangle$ satisfies the Cauchy-Schwartz inequality (Eq. 3.100) if, and only if, C_μ satisfies

$$C_\mu \leq \frac{2/3}{\mathcal{S}k/\varepsilon}, \tag{10.51}$$

or, equivalently,

$$C_\mu \leq \frac{4/9}{\mathcal{P}/\varepsilon}. \tag{10.52}$$

Show that Eq. (10.50) also holds for a general flow.

10.4.2 Model Equation for ε

Quite different approaches are taken in developing the modelled transport equations for k and ε. The k equation amounts to the exact equation (Eq. 10.35) with the turbulent flux $\mathbf{T}'$ modelled as gradient diffusion (Eq. 10.40). The three other terms—$\bar{\mathrm{D}}k/\bar{\mathrm{D}}t$, $\mathcal{P}$ and ε—are in closed form (given the turbulent viscosity hypothesis).

The exact equation for ε can also be derived, but it is not a useful starting point for a model equation. This is because (as discussed in Chapter 6) ε is best viewed as the energy flow rate in the cascade, and it is determined by the large scale motions, independent of the viscosity (at high Reynolds number). In contrast, the exact equation for ε pertains to processes in the dissipative range. Consequently, rather than being based on the exact equation, the standard model equation for ε is best viewed as being entirely empirical: it is

$$\frac{\bar{\mathrm{D}}\varepsilon}{\bar{\mathrm{D}}t} = \nabla \cdot \left(\frac{\nu_T}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{10.53}$$

The standard values of all the model constants due to Launder and Sharma (1974) are:

$$C_\mu = 0.09,\ C_{\varepsilon 1} = 1.44,\ C_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3. \tag{10.54}$$

An understanding of the ε equation can be gained by studying its behavior in different flows. We first examine homogeneous turbulence, for which the k and ε equations become

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{P} - \varepsilon, \tag{10.55}$$

and

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = C_{\varepsilon 1}\frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2}\frac{\varepsilon^2}{k}. \tag{10.56}$$

Decaying Turbulence. In the absence of mean velocity gradients, the production is zero, and the turbulence decays. The equations then have the solutions

$$k(t) = k_0\left(\frac{t}{t_0}\right)^{-n}, \quad \varepsilon(t) = \varepsilon_0\left(\frac{t}{t_0}\right)^{-(n+1)}, \tag{10.57}$$

where k and ε have the values k_0 and ε_0 at the reference time

$$t_0 = n\frac{k_0}{\varepsilon_0}, \tag{10.58}$$

and the decay exponent n is

$$n = \frac{1}{C_{\varepsilon 2} - 1}. \tag{10.59}$$

This power law decay is precisely that observed in grid turbulence (see Section 5.4.6, Eqs. 5.274 and 5.277), and so the behavior of the ε equation is correct for this flow.

The experimental values reported for the decay exponent n are generally in the range 1.15–1.45, and Mohamed and LaRue (1990) suggest that most of the data are consistent with $n = 1.3$. Equation (10.59) can be rearranged to give $C_{\varepsilon 2}$ in terms of n :

$$C_{\varepsilon 2} = \frac{n+1}{n}, \tag{10.60}$$

and corresponding to $n = 1.15$, 1.3 and 1.45, the values of $C_{\varepsilon 2}$ are 1.87, 1.77 and 1.69. It may be seen then, that the standard value ($C_{\varepsilon 2} = 1.92$) lies somewhat outside the experimentally-observed range. The reason for this is discussed below.

Exercise 10.6 Consider the k-ε model applied to decaying turbulence. Let $s(t)$ be the normalized time defined by

$$s(t) = \int_{t_0}^{t} \frac{\varepsilon(t')}{k(t')}\,\mathrm{d}t'.$$

(a) Obtain an explicit expression for $s(t)$.

(b) Derive and solve evolution equations in s for k and ε (i.e., $\mathrm{d}k/\mathrm{d}s = \ldots$).